Discrete spectrum in a critical coupling case of Jacobi matrices with spectral phase transitions by uniform asymptotic analysis

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Abstract

For a two-parameter family of Jacobi matrices exhibiting first-order spectral phase transitions, we prove discreteness of the spectrum in the positive real axis when the parameters are in one of the transition boundaries. To this end we develop a method for obtaining uniform asymptotics, with respect to the spectral parameter, of the generalized eigenvectors. Our technique can be applied to a wide range of Jacobi matrices.

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1. Introduction

In the Hilbert space $l_2(\mathbb{N})$, consider the operator J whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is the Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $\{b_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ and $\{q_n\}_{n=1}^{\infty} \subset \mathbb{R}$ (see [2, Sec. 47] for the definition of the matrix representation of an unbounded symmetric operator).

The spectral properties of the Jacobi operator J are related with the asymptotic behavior of the solutions of the following second order difference system

$$b_{n-1}f_{n-1}(\lambda) + q_n f_n(\lambda) + b_n f_{n+1}(\lambda) = \lambda f_n(\lambda), \quad n > 1, \quad \lambda \in \mathbb{R}. \tag{1.2}$$

If a solution $\{f_n(\lambda)\}_{n=1}^{\infty}$ of (1.2) also satisfies

$$q_1 f_1(\lambda) + b_1 f_2(\lambda) = \lambda f_1(\lambda) \tag{1.3}$$

then it is a generalized eigenvector of J. If, additionally, it turns out that $\{f_n(\lambda)\}_{n=1}^{\infty}$ is in $l_2(\mathbb{N})$, then this solution is an eigenvector of J^* and λ is its corresponding eigenvalue. The generalized eigenvector obtained by setting $f_1(\lambda) \equiv 1$ is the sequence of so-called polynomials of the first kind associated to the Jacobi operator J [1, Sec. 2.1 Chap. 1].

In this work we present a method for finding asymptotic expansions of solutions of (1.2) as $n \to \infty$ which gives conditions for a uniform, with respect to λ, estimate of the asymptotic remainder. This uniform asymptotic method is the main result of the present work.

In the asymptotic analysis of the solutions of (1.2), the question on uniformity with respect to λ is particularly subtle and difficult. At the same time this question arises in various applications and has been addressed before. A uniform asymptotic analysis for differential equations was carried out in [3], while the case of difference equations was treated in [26]. The results in [3] and [26] were obtained by extending to the uniform case Levinson type theorems for systems of differential equations [7, 10] and difference equations [5, 16], respectively.

The uniform generalizations of Levinson type theorems for differential and difference linear systems cannot be applied when the systems are in the so-called "double root" case (see [19]). For difference equations, the double root case corresponds to the product of transfer matrices tending to a Jordan box.

It turns out that (1.2) is in the double root case whenever the corresponding Jacobi operator exhibits first-order spectral phase transitions. An operator is said to present first-order spectral phase transition when, by variations of parameters (phases), the spectrum of the operator changes from absolutely continuous spectrum to discrete spectrum or vice versa (cf. [8, 17, 28]). It is well known that the asymptotic analysis of solutions of difference and differential equations becomes elusive and complex in the double root case, the more so when we have an equation in the presence of a parameter, as (1.2). Our uniform asymptotic approach has been concocted precisely for difference equations in the double root case.

The asymptotic method proposed here is based on an asymptotic technique due to Kelley [21] and further developed by J. Janas [15]. The majorant-minorant technique of [21] allows the asymptotic analysis of difference equations in the double root case by making use of sequences estimating solutions of a Riccati-like difference equation derived from (1.2). We extend this approach to consider the parametric difference equation (1.2) and establish conditions for uniform, with respect to the spectral parameter λ, asymptotic behavior of the solutions. We want to comment that we could not find an analogue for our uniform asymptotic method in the theory of differential equations. Remarkably, other methods for asymptotic analysis of solutions of difference equations were adapted from methods in the theory of differential equations, in particular all Levinson type theorems.

The uniform asymptotics of solutions of linear differential systems depending on a parameter was used in [4] for the spectral analysis of differential operators. In [27], the uniform asymptotic behavior of the generalized eigenvectors is pivotal in the proof of discreteness of the spectrum for a class of Jacobi matrices. In fact, the spectral analysis of operators is closely related to the *uniform* asymptotic analysis of the solutions of (1.2). Indeed, having a *pointwise* in λ asymptotics of the the solutions of (1.2), one can determine, by Subordinacy theory [13, 22], the different parts of the spectrum: absolutely continuous, singular continuous, and pure point. If one, furthermore, has *uniform* estimates of the asymptotic remainder, it is also possible to determine existence or absence of accumulation points in intervals of the pure point part of the spectrum. Moreover, the uniform asymptotic behavior of generalized eigenvectors may be used to obtain estimates for the rate of accumulation of eigenvalues at the boundaries of the pure point spectrum. We plan to address this last topic in a forthcoming paper. It is worth noting that uniform asymptotics of generalized eigenvectors is useful not only for analysis of the pure point spectrum, but for other parts as well [4].

When the asymptotic behavior of the generalized eigenvectors exhibits certain uniformity with respect to the spectral parameter, one may rule out accumulation points in the pure point part of the spectrum by recurring to ideas in [12, 9]. This was the approach in [27] for establishing discreteness of the

spectrum.

Surely there are methods for proving discreteness of the spectrum which do not use uniform asymptotic analysis of the generalized eigenvectors. One can use for instance perturbation theory, primarily Weyl theorem [20], and analysis of the behavior of the operator's quadratic form. However these "classical" methods cannot be implemented in many cases: on the one hand perturbation methods are not so useful when the unperturbed operator has a complicated form, on the other hand, if there are lacunae of pure point spectrum in the continuous spectrum the quadratic form techniques become very difficult to apply, specially when we have more than one lacuna. The uniform asymptotic approach for ruling out accumulation points in the pure point spectrum is local with respect to the spectral parameter. This makes the existence of various lacunae irrelevant for the applicability of the method.

The uniform asymptotic method proposed here is introduced by applying it to (1.2) with particular sequences $\{b_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$. We choose a class of Jacobi matrices so that (1.2) is in the double root case, which implies that the uniform asymptotic analysis cannot be carried out on the basis of uniform Levinson type theorems. In spite of the fact the we apply our method to a particular example, it will be clear from what follows that our approach has a general character and may be applied in *critical hyperbolic* cases whenever the transfer matrix entries admit an asymptotic expansion in fractional powers of $1/n$. Here, as usually in the context of differential and difference equations, the hyperbolic case implies the existence of increasing and decreasing solutions.

Let us briefly outline, step by step, the method for the uniform asymptotic analysis of (1.2).

1. **Poincaré type equations**. From the difference equation (1.2), we obtain two Poincaré type equations with smooth, with respect to n, coefficients (see (2.8) and (2.9) below). This step is straightforward, however we should remark that, in our case, the requirement for the Poincaré coefficients to be smooth yields a system of two second-order difference equations (cf. [18, 28]). The asymptotic analysis of (1.2) is carried out through the analysis of *one* of the obtained Poincaré type equations since the asymptotic behavior of the solutions of one equation determines straightforwardly the asymptotic behavior of the other.

2. **Riccati difference equation**. We derive a Riccati difference equation (see (2.14) below) from one of the Poincaré type equations. This is done by a change of variable as in [21] which is a particular realization of the so-called Riccati transformations (cf. [11, Sec. 7.2]). These transformations are well known in the continuous case (cf. [23, Sec. 6.1]).

3. **Formal asymptotic expansion**. We give an iterative procedure that provides the heuristic for obtaining a formal asymptotic expansion of two

solutions $\{X_n^{\pm}(\lambda)\}$ of (2.14). This step is an important component of the method since the next steps are based on the formal asymptotics found here.

4. **Uniform majorant and minorant sequences**. By the introduction of a new parameter in one term of the formal asymptotic expansions of $\{X_n^{\pm}(\lambda)\}$, we explicitly construct new parametrized sequences. Then we give conditions on the parameters so that the parametrized sequences serve as majorant and minorant sequences satisfying the hypothesis of Propositions 2.1 and 2.2. All this works under the requirement that the terms of the asymptotic expansion of the Poincaré coefficients, up to a certain order, are "differentiable" with respect to n. Thus, in this step we also obtain how precise our calculations for the asymptotic expansion of the Poincaré coefficients should be.

5. **Uniform asymptotics of solutions of the Riccati equation**. Using the majorant and minorant sequences found in the previous step, we obtain uniform estimating sequences for $\{X_n^{\pm}(\lambda)\}$ by applying the straightforward generalizations of Kelley's theorems, namely Propositions 2.1 and 2.2. The uniform estimating sequences allow us to prove the asymptotic formulae for $\{X_n^{\pm}(\lambda)\}$ with a uniform estimate of the asymptotic remainder.

6. **Uniform asymptotics of the solutions of the generalized eigenvectors**. From the uniform asymptotic formulae for $\{X_n^{\pm}(\lambda)\}$, we obtain the uniform asymptotic behavior of a pair of solutions of the Poincaré equation (2.8). With this information we found the uniform asymptotic expansion of two linearly independent solutions of (1.2).

Having the asymptotics of the solutions of (1.2), we make use of Subordinacy theory to prove pure point spectrum on $\mathbb{R}_+$. Then, on the basis of the *uniform* asymptotics of the generalized eigenvectors, and their smoothness with respect to λ, we show that there are no accumulation points in the pure point part of the spectrum, excluding its boundaries. This fact is proven by the technique used in [27].

We stress the fact that, for the class of Jacobi matrices discussed here, the classical methods for proving discreteness of the spectrum mentioned above have proven somehow difficult to apply because of the form of the unperturbed operator. It is also worth remarking that the spectral analysis of operators associated with these matrices may be relevant in itself. Our conclusions here shed light on the spectral properties of a two-parameter family of Jacobi operators exhibiting a first-order spectral phase transition. We prove discreteness of the spectrum in the positive real axis when the parameters are in one of the transition boundaries.

The paper is organized as follows. In Section 2 we lay down the notation, introduce the two-parameter family of Jacobi operators, and present some preparatory facts. Here we take the first two steps of the outline above. Section 3 presents step 3. In Section 4 we carry out steps 4 and 5. Section 5 contains step 6. In Section 6 we give the spectral characterization of the class of Jacobi operators under consideration. Finally, the details of calculations for the asymptotic of some concrete sequences can be found in the Appendix.

2. Preliminaries

In this section we introduce the notation, a two-parameter family of Jacobi operators, and some preliminary facts. In particular, we present the main difference equations whose asymptotic analysis is carried out in subsequent sections.

Notation. 1. Throughout this paper a sequence of numbers, depending on a real parameter λ and enumerated from some $N \in \mathbb{N}$, will be denoted by $\{g_n(\lambda)\}_{n=N}^{\infty}$. In general, N plays no rôle in a discussion focused on the asymptotic behavior of $\{g_n(\lambda)\}_{n=N}^{\infty}$ as $n \to \infty$. Nevertheless, since our goal are asymptotic expansions of $\{g_n(\lambda)\}_{n=N}^{\infty}$, uniform with respect to λ, we should watch over N and its possible dependence on λ. Having said this, we shall sometimes write $\{g_n(\lambda)\}$ instead of $\{g_n(\lambda)\}_{n=N}^{\infty}$, whenever no confusion is likely to arise.

2. Along with the standard notation of number sets, $\mathbb{N}$, $\mathbb{R}$, we use $\mathbb{R}_+$ to denote the set of real numbers greater than zero.

3. Consider a set $I \subset \mathbb{R}$, a sequence $\{g_n(\lambda)\}$, depending on a real parameter λ, and a sequence $\{h_n\}$ of real numbers. We shall say that

$$g_n(\lambda) = \widetilde{O}_I(h_n) \quad \text{as } n \to \infty$$

if there exists a constant $C > 0$ and $N \in \mathbb{N}$ such that

$$\sup_{\lambda \in I} |g_n(\lambda)| < C\,|h_n|\,, \qquad n > N\,.$$

4. Clearly, in the previous item, the constants N and C are supposed not to depend on $\lambda \in I$. There and *in the sequel auxiliary constants are assumed to be independent of* λ, unless we indicate the dependency explicitly.

5. Let $I \subset \mathbb{R}$, $\{g_n(\lambda)\}$ be a sequence depending on a real parameter λ, and $\{h_n\}$ be a sequence of real numbers. If for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\frac{\sup_{\lambda \in I} |g_n(\lambda)|}{|h_n|} < \epsilon\,, \qquad n > N\,,$$

then we say that

$$g_n(\lambda) = \widetilde{o}_I(h_n) \quad \text{as } n \to \infty \,.$$

Let us now introduce the class of operators for which we provide a uniform asymptotic analysis of the generalized eigenvectors.

In the Hilbert space $l_2(\mathbb{N})$, we define the Jacobi operator J as the one whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is (1.1) (we refer to [2, Sec. 47] for a discussion on matrix representation of unbounded symmetric operators), where the sequences $\{b_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ are defined by

$$\begin{aligned} &b_n := n^\alpha c_n\,, \qquad q_n := n^\alpha \qquad n \in \mathbb{N}\,, \\ &c_{2n-1} = c_1\,, \qquad c_{2n} = c_2\,, \qquad c_1, c_2 \in \mathbb{R}\setminus\{0\}\,, \end{aligned} \tag{2.1}$$

with

$$\alpha \in (1/3, 1/2)\,. \tag{2.2}$$

This particular choice of α is made for simplifying some calculations. We could have take $0 < \alpha < 1$, but then the derivations of some asymptotic formulae would have been hindered by algebraic technicalities by no means important for our considerations.

Allowing c_1, c_2 to vary through $\mathbb{R}\setminus\{0\}$, one obtains a two-parameter family of Jacobi operators $J = J(c_1, c_2)$. Due to the Carleman criterion [6, Thm. 1.3 Chap. 7], (2.1) implies that J is self-adjoint for any $c_1, c_2 \in \mathbb{R}\setminus\{0\}$.

As it was shown in [28] for the case $\alpha = 1$, one can immediately conclude, on the basis of results on periodically modulated Jacobi matrices [18], that J exhibits a first-order spectral phase transition for all $\alpha \in (0, 1]$. There is a region, $\left|\frac{c_1^2+c_2^2-1}{c_1c_2}\right| < 2$, where the spectrum of J, henceforth denoted by $\sigma(J)$, is purely absolutely continuous and it covers the whole real line. In the region $\left|\frac{c_1^2+c_2^2-1}{c_1c_2}\right| > 2$ the spectrum is discrete, that is, $\sigma(J) = \sigma_{disc}(J)$ (see Figure 1).

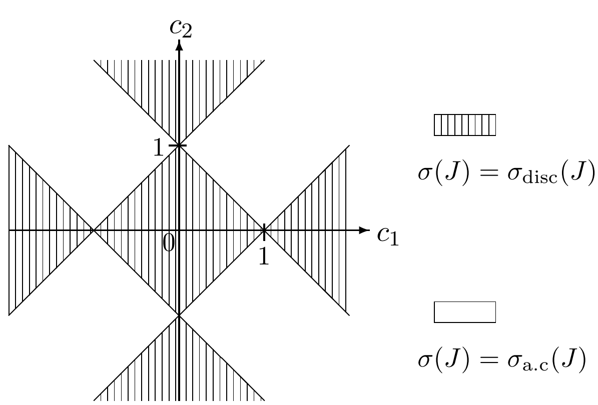

Figure 1: Phase space

The symmetry of Figure 1 obeys the existence of unitary mappings that transform J symmetrically for one quadrant of the plane (c_1, c_2) to another (see [29, Lem. 1.6]). Thus, the study of the interplay between the spectral properties of J and the coefficients c_1, c_2 can be constrained to the case

$$c_1,\, c_2 > 0\,. \tag{2.3}$$

The transition boundary $\left|\frac{c_1^2+c_2^2-1}{c_1c_2}\right| = 2$ may be split into two different cases by the conditions

$$c_1 + c_2 = 1 \tag{2.4}$$

$$|c_1 - c_2| = 1\,. \tag{2.5}$$

The spectral properties of J in the case (2.4) can be revealed by means of the asymptotic techniques used in [28] and Subordinacy theory [13, 22]. The conclusion is that, for $\alpha \in (0, 1)$ and all c_1, c_2 satisfying (2.3) and (2.4), the spectrum of J is absolutely continuous on $\mathbb{R}_+$ and discrete on $\mathbb{R}_-$ (see Figure 2). One excludes the possibility of accumulation points in the pure point part of the spectrum by repeating the reasoning of [28] which relies on estimates of the quadratic form of J and a theorem due to Glazman [14, Sec. 3 Thm. 6]. We draw the reader's attention to the fact that in the case $\alpha = 1$, with c_1, c_2 such that (2.3) and (2.4) holds, the spectrum of J is purely absolutely continuous on $(\frac{1}{2}, +\infty)$ and discrete on $(0, \frac{1}{2})$ [28, Cor. 3.4].

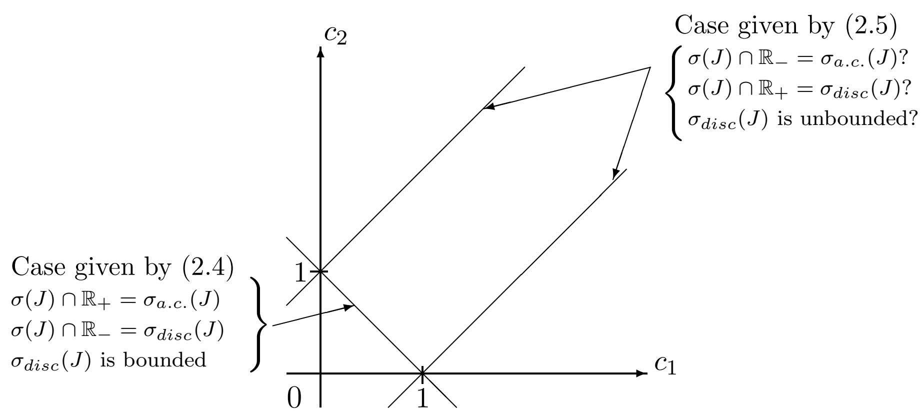

Figure 2: Transition edges

The asymptotic analysis of (1.2) will be based on a modification of Kelley's method for computing asymptotic approximations to solutions of difference equations. The starting point is the following elementary assertion by which we derive from (1.2) Poincaré type equations (cf. [11, Sec. 8.2] and the original works [24, 25]) with smooth in n coefficients. An analogous assertion can be found in [28].

Lemma 2.1. *Let $\{b_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ and $\{q_n\}_{n=1}^{\infty} \subset \mathbb{R}$. Assume that there is a number $N \in \mathbb{N}$ and a set $I \subset \mathbb{R}$ such that $q_{2n} \notin I$ for all $n > N$. For $n \geq N$ and $\lambda \in I$, define*

$$\begin{aligned} F_n(\lambda) &:= \frac{(q_{2n+2}-\lambda)b_{2n}^2}{(q_{2n}-\lambda)b_{2n+1}b_{2n+2}} - \frac{(q_{2n+1}-\lambda)(q_{2n+2}-\lambda)}{b_{2n+1}b_{2n+2}} + \frac{b_{2n+1}}{b_{2n+2}}\,, \\ G_n(\lambda) &:= \frac{(q_{2n+2}-\lambda)b_{2n-1}b_{2n}}{(q_{2n}-\lambda)b_{2n+1}b_{2n+2}}\,, \end{aligned} \tag{2.6}$$

Then, for $\lambda \in I$, the sequence $\{f_n(\lambda)\}_{n=2N}^{\infty}$ satisfies (1.2) for $n > 2N$ iff the sequences $\{x_n(\lambda)\}_{n=N}^{\infty}$ and $\{y_n(\lambda)\}_{n=N}^{\infty}$ given by

$$x_n(\lambda) := f_{2n+1}(\lambda)\,, \qquad y_n(\lambda) := f_{2n}(\lambda)\,, \quad \lambda \in I \tag{2.7}$$

satisfy the following Poincaré type equations

$$x_{n+1}(\lambda) + F_n(\lambda)x_n(\lambda) + G_n(\lambda)x_{n-1}(\lambda) = 0\,, \qquad \lambda \in I\,,\ n > N\,, \tag{2.8}$$

$$y_{n+1}(\lambda) + F_{n-\frac{1}{2}}(\lambda)y_n(\lambda) + G_{n-\frac{1}{2}}(\lambda)y_{n-1}(\lambda) = 0\,, \qquad \lambda \in I\,,\ n > N\,. \tag{2.9}$$

and the conditions

$$b_{2n-2}y_{n-1}(\lambda) + q_{2n-1}x_{n-1}(\lambda) + b_{2n-1}y_n(\lambda) = \lambda x_{n-1}(\lambda) \tag{2.10}$$

$$b_{2n-1}x_{n-1}(\lambda) + q_{2n}y_n(\lambda) + b_{2n}x_n(\lambda) = \lambda y_n(\lambda)\,, \tag{2.11}$$

for all $n > N$.

Proof. Write down (1.2) with $n = 2n, 2n+1, 2n+2$. From the equations with $n = 2n+1, 2n+2$ express $f_{2n}(\lambda)$ in terms of $f_{2n+1}(\lambda)$ and $f_{2n+3}(\lambda)$. Substitute this into the equation with $n = 2n$. It is now straightforward to verify that $\{x_n(\lambda)\}$ satisfies (2.8) Analogously, one proves that $\{y_n(\lambda)\}$ satisfies (2.9) by considering (1.2) for $n = 2n-1, 2n, 2n+1$, then expressing $f_{2n-1}(\lambda)$ through $f_{2n}(\lambda)$ and $f_{2n+2}(\lambda)$ and substituting this into the equation with $n = 2n-1$. $\square$

The sequences $\{F_n(\lambda)\}$, $\{G_n(\lambda)\}$ will be called the Poincaré coefficients.

Remark 1. Let I be any bounded interval of $\mathbb{R}_+$. If the sequences $\{b_n\}$ and $\{q_n\}$ are given by (2.1) then the conditions of Lemma 2.1 are satisfied once we choose a natural number $N \geq \frac{(\sup I)^{1/\alpha}}{2}$.

Remark 2. Note that the Poincaré coefficients defined in (2.6) are smooth in n when the sequences $\{b_n\}$ and $\{q_n\}$ are given by (2.1) and (2.5). Of course, one could have obtained directly from (1.2) a single Poincaré type equation, but, for our choice of $\{b_n\}$ and $\{q_n\}$, the corresponding Poincaré coefficients would not have been smooth in n.

Remark 3. Any solution $\{f_n(\lambda)\}_{n=1}^{\infty}$ of the three term recurrence equation (1.2) is uniquely determined by any two consecutive elements of the sequence $\{f_n(\lambda)\}_{n=1}^{\infty}$ for each λ. The same is true for solutions $\{x_n(\lambda)\}_{n=N}^{\infty}$ and $\{y_n(\lambda)\}_{n=N}^{\infty}$ of (2.8) and (2.9), respectively. Moreover, assuming that $N \in \mathbb{N}$ and $I \subset \mathbb{R}$ satisfy the conditions of Lemma 2.1, it is easy to show from (2.11) that two consecutive elements of $\{x_n(\lambda) = f_{2n+1}\}_{n=N}^{\infty}$ uniquely determine any solution $\{f_n(\lambda)\}_{n=2N+1}^{\infty}$ of (1.2) for $n > 2N + 1$.

The following straightforward assertion shows how to obtain a Riccati difference equation from a Poincaré type equation.

Lemma 2.2. *Consider (2.8) with arbitrary coefficients $\{F_n(\lambda)\}$, $\{G_n(\lambda)\}$ depending on a parameter $\lambda \in I$. Let us suppose that one can find a natural number N such that*

$$\begin{aligned} x_n(\lambda) &\neq 0\,, \qquad \lambda \in I\,, \quad n > N-1 \\ F_n(\lambda) &\neq 0\,, \qquad \lambda \in I\,, \quad n \geq N-1\,. \end{aligned}$$

For $n > N$ and $\lambda \in I$ define new coefficients

$$\beta_n(\lambda) := \frac{4G_n(\lambda)}{F_n(\lambda)F_{n-1}(\lambda)} - 1\,, \tag{2.12}$$

and a new variable

$$X_n(\lambda) := \frac{-2x_{n+1}(\lambda)}{F_n(\lambda)x_n(\lambda)} - 1\,. \tag{2.13}$$

Then, $\{x_n(\lambda)\}_{n=N}^{\infty}$ satisfies (2.8) for $n > N$ and $\lambda \in I$ iff the new sequence $\{X_n(\lambda)\}_{n=N}^{\infty}$ satisfies the Riccati difference equation

$$X_n(\lambda) = (1+\beta_n(\lambda))\frac{X_{n-1}(\lambda)}{X_{n-1}(\lambda)+1} - \beta_n(\lambda)\,, \qquad \lambda \in I\,, \quad n > N\,. \tag{2.14}$$

Proof. Consider the sequence $\{\xi_n(\lambda)\}_{n=N}^{\infty}$, whose elements are given by

$$\xi_n(\lambda) := x_n(\lambda) \prod_{k=N-1}^{n-1} \frac{-2}{F_k(\lambda)}\,, \qquad \lambda \in I \tag{2.15}$$

Substituting this expression into (2.8) for $n > N$ (cf. [11, Sec. 8.5]) , one arrives at

$$\xi_{n+1}(\lambda) - 2\xi_n(\lambda) + \frac{4G_n(\lambda)}{F_n(\lambda)F_{n-1}(\lambda)}\xi_{n-1}(\lambda) = 0\,, \quad \lambda \in I\,,\ n > N\,. \tag{2.16}$$

Taking into account that $X_n(\lambda) = \frac{\xi_{n+1}(\lambda)}{\xi_n(\lambda)} - 1$, one easily obtains that (2.16) is equivalent to (2.14) (cf. [21]). □

Remark 4. Let $\{b_n\}$, $\{q_n\}$ be defined by (2.1). On the basis of Remark 1 one may consider (2.8), with $\{F_n(\lambda)\}$ and $\{G_n(\lambda)\}$ given by (2.6), for $N_1 > \frac{(\sup I)^{1/\alpha}}{2}$ and I being a bounded interval of $\mathbb{R}_+$. If (2.5) holds and there is $N_2 \geq N_1$ such that $x_n(\lambda) \neq 0$ for all $n \geq N_2$ and $\lambda \in I$, then the conditions of Lemma 2.2 are satisfied for a certain $N \geq N_2$. This follows straightforwardly from the uniform asymptotic formula (A.1) for $\{F_n(\lambda)\}$ as $n \to \infty$.

We shall use the following elementary but important results. They are the uniform counterparts of [21, Thm. 1] and [21, Thm. 2].

Proposition 2.1. *Let I be a subset of $\mathbb{R}$. Suppose that one can find $N \in \mathbb{N}$ and real sequences $\{v_n(\lambda)\}$ and $\{w_n(\lambda)\}$ such that*

$$\inf_{\lambda\in I} v_n(\lambda) > -1\,, \qquad n \geq N \tag{2.17}$$

$$w_N(\lambda) \geq v_N(\lambda)\,, \qquad \lambda \in I\,, \tag{2.18}$$

and, for all $n > N$,

$$0 \leq 1 + \beta_n(\lambda)\,, \quad \lambda \in I \tag{2.19}$$

$$v_n(\lambda) \leq \frac{(1+\beta_n(\lambda))v_{n-1}(\lambda)}{1+v_{n-1}(\lambda)} - \beta_n(\lambda)\,, \quad \lambda \in I \tag{2.20}$$

$$w_n(\lambda) \geq \frac{(1+\beta_n(\lambda))w_{n-1}(\lambda)}{1+w_{n-1}(\lambda)} - \beta_n(\lambda)\,, \quad \lambda \in I\,. \tag{2.21}$$

Suppose that $\{X_n(\lambda)\}_{n=N}^{\infty}$ satisfies (2.14) for $n > N$ and $\lambda \in I$. Moreover, let $X_N(\lambda) \in [v_N(\lambda), w_N(\lambda)]$ for all $\lambda \in I$, then

$$v_n(\lambda) \leq X_n(\lambda) \leq w_n(\lambda)\,, \qquad n \geq N\,, \quad \lambda \in I\,.$$

Proof. The statement follows straightforwardly from the proof of [21, Thm. 1]. Here we repeat almost verbatim the proof of [21, Thm. 1], but consider all the sequences to be depending on the parameter $\lambda \in I$.

Let $n > N$ and suppose that

$$v_{n-1}(\lambda) \leq X_{n-1}(\lambda) \leq w_{n-1}(\lambda)\,, \qquad \lambda \in I \tag{2.22}$$

where $\{X_n(\lambda)\}$ is a solution of (2.14), and $\{v_n(\lambda)\}$ and $\{w_n(\lambda)\}$ satisfy (2.20) and (2.21), respectively. Due to (2.17), we have by simple algebraic calculations that, for any $\lambda \in I$,

$$\frac{v_{n-1}(\lambda)}{1+v_{n-1}(\lambda)} \leq \frac{X_{n-1}(\lambda)}{1+X_{n-1}(\lambda)} \leq \frac{w_{n-1}(\lambda)}{1+w_{n-1}(\lambda)}\,.$$

Multiplying these inequalities by $1+\beta_n(\lambda)$, we obtain in virtue of (2.19), (2.20)

and (2.21), that

$$v_n(\lambda) \le X_n(\lambda) \le w_n(\lambda)\,, \qquad \lambda \in I\,.$$

The proof is completed by induction. □

Note that inside the proof of the previous proposition we justified that $v_n(\lambda) \le w_n(\lambda)$ for all $n \ge N$.

We shall use Proposition 2.1 to obtain the uniform asymptotics of an increasing solution of the Riccati difference equation (2.14). To obtain a uniform asymptotic expansion for a decreasing solution of (2.14), one has to use the following result.

Proposition 2.2. *Let I be a subset of $\mathbb{R}$. Suppose that one can find $N \in \mathbb{N}$ and real sequences $\{v_n(\lambda)\}$ and $\{w_n(\lambda)\}$ such that*

$$v_n(\lambda) \ge w_n(\lambda)\,, \qquad \lambda \in I\,, \quad n \ge N\,, \tag{2.23}$$

$$\sup_{\substack{\lambda\in I\\ n\ge N}} |v_n(\lambda)|\,, \sup_{\substack{\lambda\in I\\ n\ge N}} |w_n(\lambda)| < 1\,, \tag{2.24}$$

and (2.19)–(2.21) hold for $n > N$. Then (2.14) has a solution $\{X_n(\lambda)\}_{n=N}^{\infty}$ satisfying

$$v_n(\lambda) \ge X_n(\lambda) \ge w_n(\lambda)\,, \qquad n \ge N\,, \quad \lambda \in I\,. \tag{2.25}$$

Proof. The proof of this assertion follows the proof of [21, Thm. 2], taking into account the dependence on $\lambda \in I$.

Let us fix a natural number $s > N$ and define

$$X_{n,s}(\lambda) := w_n(\lambda)\,, \qquad n \ge s\,. \tag{2.26}$$

By using recurrently the formula

$$X_{n-1,s}(\lambda) := \frac{X_{n,s}(\lambda) + \beta_n(\lambda)}{1 - X_{n,s}(\lambda)}\,, \tag{2.27}$$

for $n = s, s-1, \dots, N+1$, one defines $X_{n,s}(\lambda)$ for all $n \ge N$. Clearly the sequence $\{X_{n,s}(\lambda)\}_{n=N}^{\infty}$ satisfies (2.14) for $N < n \le s$.

The inequalities (2.20) and (2.24) imply

$$v_{n-1}(\lambda) \ge \frac{v_n(\lambda) + \beta_n(\lambda)}{1 - v_n(\lambda)}\,, \qquad n > N\,. \tag{2.28}$$

Analogously, from (2.21) and (2.24), one obtains

$$w_{n-1}(\lambda) \le \frac{w_n(\lambda) + \beta_n(\lambda)}{1 - w_n(\lambda)}\,, \qquad \lambda \in I\,, \quad n > N\,. \tag{2.29}$$

It follows from (2.27), (2.28), and (2.29) that

$$w_n(\lambda) \leq X_{n,s}(\lambda) \leq v_n(\lambda)\,, \qquad \lambda \in I\,, \tag{2.30}$$

for $N \leq n < s$. But, due to (2.26) and (2.23), the inequality (2.30) actually holds for $n \geq N$.

Now, fix a natural number $n \geq N$ and consider the sequence $\{X_{n,s}(\lambda)\}_{s=s_0}^{\infty}$, with $s_0 > N$. On the basis of (2.29), it can be verified that $\{X_{n,s}(\lambda)\}_{s=s_0}^{\infty}$ is a monotone non-decreasing function and, by (2.30), it is bounded from above. Then the sequence has a limit that we denote by $X_n(\lambda)$. This can be done for any $n \geq N$. Thus, we have constructed a sequence $\{X_n(\lambda)\}_{n=N}^{\infty}$ which satisfies (2.14) and is such that (2.25) holds. $\square$

3. Formal asymptotic analysis of the Riccati equation

In this section we present a heuristic approach for obtaining formal asymptotic expansions of solutions of the Riccati difference equation (2.14). The asymptotics is constructed term by term beginning from the leading one. The algorithm given below may be iterated multiple times until the desired precision. We remark that in this section there will not be proofs for our asymptotic formulae. Nevertheless, one may adapt our heuristic to prove *pointwise*, with respect to λ, asymptotic expansions. We are not interested in this pointwise asymptotics since our approach will yield a stronger result.

Clearly, (2.14) can be written as follows

$$X_n(\lambda) + X_n(\lambda)X_{n-1}(\lambda) = X_{n-1}(\lambda) - \beta_n(\lambda) \tag{3.1}$$

This is the starting point of the algorithm and the first step is to find the leading term of the asymptotic expansion as $n \to \infty$ of $\{X_n(\lambda)\}$. To this end we assume that the sequence $\{X_n(\lambda)\}$ tend to 0 as $n \to \infty$ and that it is smooth with respect to n. Then the leading term satisfies the relation

$$X_n(\lambda) + X_n^2(\lambda) = X_n(\lambda) - \beta_n(\lambda)\,. \tag{3.2}$$

Therefore the main term of the asymptotic formula of $\{X_n(\lambda)\}$ coincides with that of the asymptotic expansion of $\{\pm\sqrt{-\beta_n(\lambda)}\}$. The next terms of the expansion may be found by introducing a rectifying sequence $\{t_n(\lambda)\}$ such that

$$X_n^2(\lambda) = -\beta_n(\lambda) + t_n(\lambda)\,. \tag{3.3}$$

Expressing $\beta_n(\lambda)$ through $X_n(\lambda)$ and $t_n(\lambda)$ from (3.3) and substituting it into (3.1), one obtains the exact equation

$$t_n(\lambda) = (X_n(\lambda) - 1)(X_n(\lambda) - X_{n-1}(\lambda))\,. \tag{3.4}$$

Since $\beta_n(\lambda) \to 0$ for each λ, the leading term of $\{t_n(\lambda)\}$ is given by the leading term of $\{\pm\left(\sqrt{-\beta_n(\lambda)} - \sqrt{-\beta_{n-1}(\lambda)}\right)\}$.

For $\{\beta_n(\lambda)\}$ given by (2.12) with $\{b_n\}$, $\{q_n\}$ defined by (2.1) and (2.5), we show in Appendix B that the uniform asymptotic expansion of $\{\beta_n(\lambda)\}$ as $n \to \infty$, when $\lambda > 0$, obeys formula (B.3). Thus, by the elementary relation

$$n^s - (n-1)^s = sn^{s-1} + O(n^{s-2}) \quad as \quad n \to \infty\,, \quad s \in \mathbb{R}\,, \tag{3.5}$$

one easily concludes that for each $\lambda > 0$

$$t_n(\lambda) = \sqrt{\lambda \frac{2^{1-\alpha}}{c_1 c_2}}\, n^{-1-\alpha/2} + o(n^{-1-\alpha/2})\,, \quad \text{as} \quad n \to \infty\,.$$

We have, therefore, found without proving that for each $\lambda > 0$

$$X_n(\lambda) = \pm\sqrt{-\beta_n(\lambda)} + \frac{\alpha}{4n} + o(n^{-1})\,, \quad \text{as} \quad n \to \infty\,. \tag{3.6}$$

To improve the precision of the previous uniform asymptotic formula one may carry out a second iteration of our heuristic reasoning. To this end one introduces a second rectifying sequence $\{u_n(\lambda)\}$ such that

$$X_n^2(\lambda) = -\beta_n(\lambda) + \sqrt{\lambda \frac{2^{1-\alpha}}{c_1 c_2}}\, n^{-1-\alpha/2} + u_n(\lambda)$$

and repeats what was done before to find an expression for $t_n(\lambda)$, namely, one expresses $\beta_n(\lambda)$ through $X_n(\lambda)$ and $u_n(\lambda)$ from the equation above and substitutes it into (3.1).

For the purpose of the present work, the precision formally obtained by the first iteration is sufficient. We remind that our formal asymptotic expansion will be used only to find the structure of the majorant and minorant sequences which are fundamental for our uniform asymptotic method.

4. Uniform asymptotic analysis of the Riccati equation: the modified Kelley method

This section is devoted to the uniform asymptotic analysis of the solutions of the Riccati equation (2.14). To this end, we construct the majorant and minorant sequences of Propositions 2.1 and 2.2. The formal asymptotics (3.6) found in the previous section and the uniform asymptotic behavior of the sequence $\{\beta_n(\lambda)\}$, given in (B.3), will be at the basis of our considerations.

After reminding the reader on the convention 4 of our notation given in Section 2, let us consider the following simple and straightforward lemmas.

Lemma 4.1. *Let $\psi_0(\lambda)$ be a uniformly positive and bounded function defined on $I \subset \mathbb{R}$, i. e.,*

$$\inf_{\lambda\in I} \psi_0(\lambda) > 0\,, \qquad \sup_{\lambda\in I} \psi_0(\lambda) < \infty\,. \tag{4.1}$$

Let $0 < p < 1$ and $M \in \mathbb{N}$. Suppose that there is a real sequence $\{\varphi_n(\lambda)\}$ having the following asymptotic behavior as $n \to \infty$

$$\varphi_n(\lambda) = \sum_{k=0}^{M} \psi_k(\lambda) n^{-s_k} + \widetilde{o}_I\left(n^{-p-1}\right)\,, \qquad \lambda \in I\,, \tag{4.2}$$

where $\psi_k(\lambda)$ is bounded for $k = 1, \dots, M$ and $p = s_0 < s_k \le p + 1$. Then, for any fixed constants $A^\pm$ obeying

$$A^+ < \frac{p}{2} < A^-\,, \tag{4.3}$$

there exists $N \in \mathbb{N}$ such that the sequences $\{v_n^\pm(\lambda)\}_{n=N}^\infty$ given by

$$v_n^\pm(\lambda) := \pm\varphi_n(\lambda) + A^\pm n^{-1}\,, \qquad \lambda \in I\,, \tag{4.4}$$

satisfy

$$v_n^\pm(\lambda)\left(1 + v_{n-1}^\pm(\lambda)\right) \le v_{n-1}^\pm(\lambda) + \varphi_n^2(\lambda)\,, \qquad n > N\,, \qquad \lambda \in I\,. \tag{4.5}$$

Proof. By substituting (4.4) into (4.5), one can show that there is a sequence $\{\zeta_n(\lambda)\}$ such that the inequality (4.5) for $\{v_n^+(\lambda)\}$ is reduced to

$$A^+\left(\frac{\varphi_{n-1}(\lambda)}{n} + \frac{\varphi_n(\lambda)}{n-1}\right) \le \left(\varphi_n(\lambda) - 1\right)\left(\varphi_n(\lambda) - \varphi_{n-1}(\lambda)\right) + \zeta_n(\lambda)\,,$$

where $\zeta_n(\lambda) = \widetilde{O}_I(n^{-2})$ as $n \to \infty$. We allow ourselves to write instead of the previous inequality that as $n \to \infty$

$$A^+\left(\frac{\varphi_{n-1}(\lambda)}{n} + \frac{\varphi_n(\lambda)}{n-1}\right) \le \left(\varphi_n(\lambda) - 1\right)\left(\varphi_n(\lambda) - \varphi_{n-1}(\lambda)\right) + \widetilde{O}_I\left(n^{-2}\right)\,, \tag{4.6}$$

hoping that it will not lead to misunderstanding.

Using (3.5), one easily obtains the uniform asymptotics

$$\varphi_n(\lambda) - \varphi_{n-1}(\lambda) = -p\psi_0(\lambda)n^{-p-1} + \widetilde{o}_I\left(n^{-p-1}\right)\,, \quad \text{as} \quad n \to \infty\,, \tag{4.7}$$

and then one verifies, after elementary calculations, that (4.6) may be reduced to

$$\psi_0(\lambda)\left(2A^+ - p\right) \le \widetilde{o}_I\left(1\right)\,, \quad \text{as} \quad n \to \infty\,, \tag{4.8}$$

for a suitable $\widetilde{o}_I(1)$ sequence. In view of (4.8) and the strict positivity of $\psi_0(\lambda)$

for $\lambda \in I$, the first inequality in (4.3), that is $A^+ < \frac{p}{2}$, ensures the existence of $N_1 \in \mathbb{N}$ such that $\{v_n^+(\lambda)\}$ fulfills (4.5) for $n > N_1$. Note that N_1 depends on what constant A^+ satisfying the first inequality in (4.3) has been chosen.

Let us now consider (4.5) for $\{v_n^-(\lambda)\}$. Calculations analogous to that leading to (4.6) here yield that (4.5) is reduced to

$$A^- \left(\frac{\varphi_{n-1}(\lambda)}{n} + \frac{\varphi_n(\lambda)}{n-1} \right) \geq (\varphi_n(\lambda) + 1)\left(\varphi_{n-1}(\lambda) - \varphi_n(\lambda)\right) + \widetilde{O}_I\left(n^{-2}\right) ,$$

as $n \to \infty$. This inequality holds if and only if, for $\lambda \in I$,

$$\psi_0(\lambda)\left(2A^- - p\right) \geq \widetilde{o}_I(1) , \quad as \quad n \to \infty . \tag{4.9}$$

From (4.9) one concludes that, when the second inequality in (4.3) holds, that is $A^- > \frac{p}{2}$, there is a number N_2 such that $\{v_n^-(\lambda)\}$ satisfies (4.5) for $n > N_2$. The number N_2 depends on the constant A^- chosen to satisfy the second inequality in (4.3). The proof is complete by setting $N = \max\{N_1, N_2\}$. □

Lemma 4.2. *Let the conditions on $\psi_k(\lambda)$ for $k = 0, 1, \ldots, M$ and $\{\varphi_n(\lambda)\}$ of Lemma 4.1 be satisfied. Then, for any fixed constants $B^\pm$ obeying*

$$B^- < \frac{p}{2} < B^+ , \tag{4.10}$$

there exists $N \in \mathbb{N}$ such that the sequences $\{w_n^\pm(\lambda)\}_{n=N}^\infty$ given by

$$w_n^\pm(\lambda) := \pm\varphi_n(\lambda) + B^\pm n^{-1} , \qquad \lambda \in I , \tag{4.11}$$

satisfy

$$w_n^\pm(\lambda)\left(1 + w_{n-1}^\pm(\lambda)\right) \geq w_{n-1}^\pm(\lambda) + \varphi_n^2(\lambda) , \qquad n > N , \qquad \lambda \in I . \tag{4.12}$$

Proof. The proof repeats the reasoning of that of Lemma 4.1. □

We draw the reader's attention to the following. First, in (4.4) and (4.11) we have reproduced the structure of the formal asymptotics of solutions of the Riccati equation (3.6). Second, if the asymptotic expansion of $\{\varphi_n(\lambda)\}$ had been given in a less precise form than (4.2), it would not have been sufficient for proving that (4.7) holds true and, then, for proving the assertions of Lemmas 4.1 and 4.2. Note that, for obtaining a condition on A^+, the order of the leading term in the right-hand side of (4.6) must coincide with that of the left-hand side. The expansion (4.7) ensures this, but for (4.7) to hold, one needs to know that all terms of the asymptotic expansion of $\{\varphi_n(\lambda)\}$ are "differentiable with respect to n" up to the precision indicated by (4.2). Clearly, it is not important to know the concrete form of the functions $\psi_k(\lambda)$, $k = 0, 1, \ldots, M$ as long as they satisfy the conditions of Lemmas 4.1 and 4.2.

Bellow we shall show that the majorant and minorant sequences referred at the beginning of this section are constructed from the sequences $\{v_n^{\pm}(\lambda)\}$, $\{w_n^{\pm}(\lambda)\}$ under the assumption that

$$\varphi_n(\lambda) = \sqrt{-\beta_n(\lambda)}\,. \tag{4.13}$$

Thus, (4.2) shows how precise our calculations of the asymptotic expansion of $\{\beta_n(\lambda)\}$, and consequently of the Poincaré coefficients, should be. Note that, by considering the remark at the end of the previous paragraph, we are not concerned with the concrete form of the corresponding functions $\psi_k(\lambda)$, $k = 1, \dots, M$, stemming from (4.13).

Lemma 4.3. *Let (2.2) and (2.3) hold and let $\{\beta_n(\lambda)\}$ have the asymptotic behavior given by (B.3). Suppose that $A^{\pm}$ and $B^{\pm}$ obey (4.3) and (4.10). Then, for any bounded subset $I \subset \mathbb{R}_+$ separated from zero, the sequences $\{v_n^{+}(\lambda)\}$, $\{w_n^{+}(\lambda)\}$ and $\{v_n^{-}(\lambda)\}$, $\{w_n^{-}(\lambda)\}$, given by (4.4) and (4.11) with (4.13), satisfy the conditions of Propositions 2.1 and 2.2, respectively.*

Proof. First note that (B.3) implies the existence of $n_1 \in \mathbb{N}$ such that

$$\inf_{\lambda\in I}\{1 + v_n^{\pm}(\lambda)\} > 0\,, \qquad \inf_{\lambda\in I}\{1 + w_n^{\pm}(\lambda)\} > 0\,, \qquad n \ge n_1\,. \tag{4.14}$$

Thus, (2.17) holds for $n \ge n_1$.

Now, in view of the asymptotic expansion (B.3), the sequence $\{\varphi_n(\lambda)\}$ satisfies the conditions required by Lemmas 4.1 and 4.2. Indeed, on the one hand, the condition on p holds due to (2.2) since $p = \frac{\alpha}{2}$. On the other hand, a simple computation shows that

$$\psi_0(\lambda) = \sqrt{\lambda\frac{2^{1-\alpha}}{c_1c_2}}\,, \qquad \lambda \in I\,.$$

Thus, taking into account that the bounded set $I \subset \mathbb{R}_+$ is separated from zero, (2.3) implies that $\psi_0(\lambda)$ satisfies (4.1). The conditions on $\psi_k(\lambda)$, $k = 1, \dots, M$, are easily verified.

By Lemmas 4.1 and 4.2 there is a natural number $n_2 \ge n_1$ such that

$$v_n^{\pm}(\lambda)\left(1 + v_{n-1}^{\pm}(\lambda)\right) \le v_{n-1}^{\pm}(\lambda) - \beta_n(\lambda)\,, \qquad n > n_2\,, \qquad \lambda \in I\,, \tag{4.15}$$
$$w_n^{\pm}(\lambda)\left(1 + w_{n-1}^{\pm}(\lambda)\right) \ge w_{n-1}^{\pm}(\lambda) - \beta_n(\lambda)\,, \qquad n > n_2\,, \qquad \lambda \in I\,. \tag{4.16}$$

Due to (4.14), the inequalities (4.15) and (4.16) are, respectively, equivalent to (2.20) and (2.21) for $n \ge n_2$. On the other hand, (B.3) implies the fulfilment of (2.19) for some n_3 independent of λ. We choose $n_3 \ge n_2$. After observing that $v_{n_3}^{+}(\lambda) < w_{n_3}^{+}(\lambda)$ for all $\lambda \in I$, one can conclude that $\{v_n^{+}(\lambda)\}$ and $\{w_n^{+}(\lambda)\}$ satisfy the conditions of Proposition 2.1 with $N = n_3$.

On the basis of (B.3), one easily concludes that there is $n_4 \in \mathbb{N}$ such that $\{v_n^-(\lambda)\}$ and $\{w_n^-(\lambda)\}$ satisfy (2.23) and (2.24), respectively, for $n \geq n_4$. Since (2.18)–(2.20) are also fulfilled for some $n_5 \geq n_4$, one verifies that $\{v_n^-(\lambda)\}$ and $\{w_n^-(\lambda)\}$ obey the conditions of Proposition 2.2 with $N = n_5$. $\square$

The following result gives the uniform asymptotics of solutions of the Riccati equation (2.14).

Theorem 4.1. *Let (2.2) and (2.3) hold, and suppose that $\{\beta_n(\lambda)\}$ has the asymptotic behavior given by (B.3). Then, for any bounded set $I \subset \mathbb{R}_+$ separated from zero, there is $N \in \mathbb{N}$ such that there are solutions $\{X_n^\pm(\lambda)\}_{n=N}^\infty$ of (2.14) for $n \geq N$ having the following asymptotic behavior*

$$X_n^\pm(\lambda) = \pm\sqrt{-\beta_n(\lambda)} + \widetilde{O}_I\left(n^{-1}\right) \qquad \text{as } n \to \infty\,. \tag{4.17}$$

Proof. Consider the sequences $\{v_n^\pm(\lambda)\}$ and $\{w_n^\pm(\lambda)\}$ given by (4.4) and (4.11). By Lemma 4.3 and Proposition 2.1 there is a constant $N_1 \in \mathbb{N}$ and a solution $\{X_n^+(\lambda)\}$ of (2.14) such that, for all $\lambda \in I$,

$$v_n^+(\lambda) \leq X_n^+(\lambda) \leq w_n^+(\lambda)\,, \qquad n \geq N_1\,.$$

Due to (4.4) and (4.11)

$$w_n^+(\lambda) - v_n^+(\lambda) = \left(B^+ - A^+\right) n^{-1}\,.$$

Similarly, by Lemma 4.3 and Proposition 2.2, one can find $N_2 \in \mathbb{N}$ and a solution $\{X_n^-(\lambda)\}$ of (2.14) such that for all $\lambda \in I$

$$v_n^-(\lambda) \geq X_2^-(\lambda) \geq w_n^-(\lambda)\,, \qquad n \geq N_2\,.$$

The proof is complete by noting that

$$v_n^-(\lambda) - w_n^-(\lambda) = \left(A^- - B^-\right) n^{-1}$$

and setting $N = \max\{N_1, N_2\}$. $\square$

5. Uniform asymptotics of the generalized eigenvectors

The results of the previous section allow us to obtain the uniform asymptotic behavior of the generalized eigenvectors corresponding to the family of Jacobi matrices $J(c_1, c_2)$ introduced in Section 2. Our results are restricted to the case given by (2.3) and (2.5) and illustrated in Figure 2.

We begin by finding the asymptotic behavior of the solutions of the Poincaré type equation (2.8).

Theorem 5.1. *Let $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be defined by (2.1) with (2.2), (2.3) and (2.5). Also let I be any bounded subset of $\mathbb{R}_+$ separated from zero. Then, there is $N \in \mathbb{N}$ such that there are solutions $\{x_n^{\pm}(\lambda)\}_{n=N}^{\infty}$ of (2.8) for $n > N$, with $\{F_n(\lambda)\}$ and $\{G_n(\lambda)\}$ given by (2.6), having the following uniform asymptotic behavior as $n \to \infty$*

$$x_n^{\pm}(\lambda) = \exp\left(\pm \frac{\sqrt{\lambda \frac{2^{1-\alpha}}{c_1 c_2}}}{1 - \frac{\alpha}{2}} n^{1-\frac{\alpha}{2}} + \widetilde{O}_I(n^{1-\alpha}) \right) . \tag{5.1}$$

Proof. Since (2.2) and (2.5) hold, the asymptotic expansion of $\{F_n(\lambda)\}$ is given by (A.1). This implies that there is $N_1 \in \mathbb{N}$ such that the conditions of Lemma 2.2 are satisfied (see Remark 4). Then the solutions of (2.8) are given by the solutions of (2.14) for $n > N_1$ and $x_n(\lambda) \neq 0$. According to (2.13), the sequence $\{x_n(\lambda)\}_{n=N_1}^{\infty}$ satisfies

$$\frac{x_{n+1}(\lambda)}{x_n(\lambda)} = (-1) \frac{F_n(\lambda)}{2} \left(1 + X_n(\lambda)\right) , \qquad n \geq N_1,$$

where $\{X_n(\lambda)\}_{n=N_1}^{\infty}$ is a solution of (2.14) and $x_n(\lambda) \neq 0$ for $n > N_1$. Hence,

$$x_n(\lambda) = x_{N_1}(\lambda)(-1)^{n-N_1} \prod_{k=N_1}^{n-1} \left[\frac{F_k(\lambda)}{2} \left(1 + X_k(\lambda)\right) \right] , \tag{5.2}$$

with $x_{N_1}(\lambda) \neq 0$ for $\lambda \in I$. Let us find the asymptotic behavior of the product. We assume that $N_2 \geq N_1$ is such that

$$\sup_{\lambda \in I} |F_n(\lambda) - 2| < \frac{1}{2} , \qquad \sup_{\lambda \in I} |X_n(\lambda)| < \frac{1}{2} \quad n \geq N_2$$

This can be done by virtue of (A.1) and (4.17) taking into account (B.3).

We have

$$\prod_{k=N_2}^{n-1} \frac{F_k(\lambda)}{2} = \exp\left(\sum_{k=N_2}^{n-1} \log\left(\frac{F_k(\lambda)}{2} \right) \right) .$$

On the basis of (A.1), one verifies that

$$\prod_{k=N_2}^{n-1} \frac{F_k(\lambda)}{2} = \exp \widetilde{O}_I(n^{1-\alpha}) , \qquad \text{as} \quad n \to \infty . \tag{5.3}$$

On the other hand, by (2.2), (2.3) and (2.5), the asymptotic expansion (B.3) holds and the conditions of Theorem 4.1 are met. Thus, there is $N_3 \geq N_2$ such

that (2.14) has solutions (4.17) for $n > N_3$. Now, in view of [21, Sec. 2],

$$\prod_{k=N_3}^{n-1} \left(1 + X_k^{\pm}(\lambda)\right) = K_n(\lambda) \exp\left(\sum_{k=N_3}^{n-1} \sum_{j=1}^{s} \frac{(-1)^{j-1}}{j} \left(X_k^{\pm}(\lambda)\right)^j \right), \tag{5.4}$$

where $K_n(\lambda) \to K(\lambda) > K > 0$, as $n \to \infty$, and s has been chosen so that $\sup_{\lambda \in I} \sum_{k=N_3}^{\infty} \left(X_k^{\pm}(\lambda)\right)^{s+1} < \infty$. We can always find such s in view of (4.17) and (B.3). Straightforward computations yield the following asymptotic behavior

$$\prod_{k=N_3}^{n-1} \left(1 + X_k^{\pm}(\lambda)\right) = \exp\left(\pm \frac{\sqrt{\lambda \frac{2^{1-\alpha}}{c_1 c_2}}}{1 - \frac{\alpha}{2}} n^{1-\frac{\alpha}{2}} + \widetilde{O}_I(n^{1-\frac{3}{2}\alpha}) \right), \qquad \text{as} \qquad n \to \infty .$$

For the proof to be complete, set $N := N_3$. $\square$

Remark 5. We could have obtained a more precise asymptotic formula than (5.1). This may be done by using an asymptotic expansion of $\{\beta_n(\lambda)\}$ more precise than (B.3) and refining the results of Section 4. However, for our goal, the precision of the asymptotics (5.1) is sufficient.

Corollary 5.1. *Let $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be defined by (2.1) with (2.2), (2.3) and (2.5). Suppose that I is a bounded subset of $\mathbb{R}_+$ separated from zero. Then, there is $N \in \mathbb{N}$ such that there are linearly independent solutions $\{f_n^{\pm}(\lambda)\}_{n=N}^{\infty}$ of (1.2) for $n > N$ having the following uniform asymptotic behavior as $n \to \infty$*

$$f_n^{\pm}(\lambda) = \exp\left(\pm \frac{\sqrt{\frac{\lambda}{2c_1 c_2}}}{1 - \frac{\alpha}{2}} n^{1-\frac{\alpha}{2}} + \widetilde{O}_I(n^{1-\alpha}) \right). \tag{5.5}$$

Proof. By Remark 3, the solution $\{x_n^{+}(\lambda)\}$, respectively $\{x_n^{-}(\lambda)\}$, of (2.8) found in Theorem 5.1 determines uniquely a solution $\{y_n^{+}(\lambda)\}$, respectively $\{y_n^{-}(\lambda)\}$, of (2.9). By (2.11) we have the following asymptotic formula as $n \to \infty$

$$y_n^{\pm}(\lambda) = \exp\left(\pm \frac{\sqrt{\lambda \frac{2^{1-\alpha}}{c_1 c_2}}}{1 - \frac{\alpha}{2}} n^{1-\frac{\alpha}{2}} + \widetilde{O}_I(n^{1-\alpha}) \right).$$

The solutions of (1.2) for $n > N$ are obtained from the sequences $\{x_n^{\pm}(\lambda)\}$ and $\{y_n^{\pm}(\lambda)\}$ by means of (2.7). Clearly, $\{f_n^{+}(\lambda)\}$ and $\{f_n^{-}(\lambda)\}$ are linearly independent and every solution of (1.2), for $n > N$, is a linear combination of these sequences. $\square$

Remark 6. The solutions $\{f_n^{\pm}(\lambda)\}_{n=N}^{\infty}$ of (1.2) for $n > N$ given by Corollary 5.1 may be extended by Remark 3 to sequences $\{f_n^{\pm}(\lambda)\}_{n=1}^{\infty}$ being linearly independent solutions of (1.2) for $n > 1$. All solutions of (1.2) for $n > 1$ are

linear combinations of $\{f_n^{\pm}(\lambda)\}_{n=1}^{\infty}$, in particular, the generalized eigenvectors of J (see the Introduction).

Corollary 5.2. *Let $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be defined by (2.1) with (2.2), (2.3) and (2.5). Suppose that I is a bounded subset of $\mathbb{R}_+$ separated from zero and that $\lambda_0 \in \sigma_{pp}(J) \cap I$. Then the sequence*

$$\{f_n^-(\lambda_0)\}_{n=1}^{\infty},$$

where $\{f_n^-(\lambda)\}_{n=1}^{\infty}$ is the sequence given in Remark 6, is the eigenvector of J corresponding to λ_0. Moreover, there is a uniformly separated from zero and bounded function

$$\mathcal{C} : \sigma_{pp}(J) \cap I \to \mathbb{R}$$

such that

$$f_n^*(\lambda) = \mathcal{C}(\lambda) f_n^-(\lambda), \quad \lambda \in \sigma_{pp}(J) \cap I, \tag{5.6}$$

where $\{f_n^*(\lambda)\}$ is the sequence of orthogonal polynomials of the first kind associated to J (see the Introduction).

Proof. Consider equation (1.2). Assign $f_1^*(\lambda) := 1$ for all λ and $f_2^*(\lambda) := \frac{\lambda - q_1}{b_1}$ By using recurrently (1.2), one defines the sequence $\{f_n^*(\lambda)\}_{n=1}^{\infty}$ of orthogonal polynomials of the first kind associated with J. Clearly, evaluation of this sequence in λ_0 yields an eigenvector.

Now, since $\lambda_0 \in \sigma_{pp}(\lambda)$ the solution $\{f_n^*(\lambda_0)\}_{n=1}^{\infty}$ of (1.2) is decaying and therefore it is equal to $\{f_n^-(\lambda_0)\}_{n=1}^{\infty}$ modulo a constant factor. From this, one obtains the first assertion of the corollary and (5.6). The stated properties of the function $\mathcal{C}$ follow from the boundedness of $f_n^*(\lambda)$ with respect to $\lambda \in I$ for and fixed n, the asymptotic formula (5.5), and the fact that $\mathcal{C}$ is independent of n. Indeed, fix a natural number n sufficiently large so that (5.5) holds, then $f_n^-(\lambda)$ is uniformly positive and bounded for $\lambda \in I$. Thus, by the boundedness of the polynomials of the first kind, we obtain that $\mathcal{C}$ is a uniformly separated from zero and bounded function for λ belonging to $\sigma_{pp}(J) \cap I$.

Note that under our considerations the roots of the polynomial $f_n^*(\lambda)$ for any sufficiently large $n \in \mathbb{N}$ are not in $\sigma_{pp}(J)$. $\square$

6. Discrete spectrum

The main results of the previous section indicate, among other things, that the sequence $\{f_n^-(\lambda)\}_{n=1}^{\infty}$, given in Remark 6, is a subordinate solution [13, 22] of the recurrence equation (1.2) for $\lambda \in I$, where $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are given by (2.1), (2.2), (2.3), (2.5), and I is a bounded subset of $\mathbb{R}_+$ separated from zero. Thus, by invoking [13, 22], we arrive at the following assertion.

Theorem 6.1. *Let the sequences $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be given by (2.1) with (2.2), (2.3), and (2.5). Then the spectrum of J in $\mathbb{R}_+$ is pure point.*

We expect the spectrum to be discrete in $\mathbb{R}$, that is finite in I, on the grounds that the uniform estimate of the asymptotic remainder allows, "in some sense", to avoid dealing with the generalized eigenvector's tail. This very informal reasoning provides a clue to the problem of estimates for the number of eigenvalues, namely by recurring to [14, Sec. 3 Thm. 6]. Below we show that this speculation on the discreteness of the spectrum is indeed true: we prove absence of accumulation points of $\sigma_{pp}(J)$ in any closed bounded interval of $\mathbb{R}_+$. To this end we make use of a technique developed in [27] which, in its turn, relies on ideas put forth in [9, 12]. The main ingredients of this recipe are the uniform asymptotic formulae found in the previous section and Corollary 5.2.

We shall need the following auxiliary result.

Lemma 6.1. *For any closed bounded interval I of $\mathbb{R}_+$ and any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that*

$$\sup_{\lambda \in I} \sum_{n=K}^{\infty} \left| f_n^-(\lambda) \right|^2 < \epsilon \,,$$

where $\{f_n^-(\lambda)\}$ is the sequence given in Remark 6.

Proof. The assertion follows straightforwardly from the uniform asymptotics (5.5). □

Theorem 6.2. *Let the sequences $\{q_n\}$ and $\{b_n\}$ be given by (2.1) with (2.2), (2.3) and (2.5). Then the spectrum of J in $\mathbb{R}_+$ is discrete, i. e.,*

$$\sigma(J) \cap \mathbb{R}_+ = \sigma_{disc}(J) \cap \mathbb{R}_+ \,.$$

Proof. By Theorem 6.1 we know that the spectrum is pure point in any closed bounded interval I of $\mathbb{R}_+$. Suppose that $\sigma(J)$ has a point of accumulation μ in the interior of I. Let λ and λ' ($\lambda \neq \lambda'$) be arbitrarily chosen from $\sigma(J) \cap V_\delta(\mu)$, where $V_\delta(\mu)$ is a δ-neighborhood of μ, and δ is so small that $V_\delta(\mu) \subset I$. Of course, μ need not be itself an eigenvalue.

For any $K \in \mathbb{N}$ the following inequality holds

$$\begin{aligned} \left| (f^*(\lambda), f^*(\lambda'))_{l_2(\mathbb{N})} \right| &= \left| \sum_{n=1}^{\infty} f_n^*(\lambda) \overline{f_n^*(\lambda')} \right| \geq \\ &\geq \left| \sum_{n=1}^{K} f_n^*(\lambda) \overline{f_n^*(\lambda')} \right| - \left| \sum_{n>K} f_n^*(\lambda) \overline{f_n^*(\lambda')} \right| , \end{aligned} \tag{6.1}$$

where $\{f_n^*(\lambda)\}$ is the sequence of polynomials of the first kind associated with J.

Let us estimate the last term in the right hand side of (6.1). To this end express $f_n^*(\lambda)$ and $f_n^*(\lambda')$ through (5.6). Now, on the basis of Corollary 5.2, namely

the boundedness of $\mathcal{C}$, we verify that Lemma 6.1 and the Cauchy-Schwartz inequality allow us to choose K independent of $\lambda, \lambda' \in I$, so that

$$\left|\sum_{n>K} f_n^*(\lambda)\overline{f_n^*(\lambda')}\right| < \frac{1}{2}\,. \tag{6.2}$$

Now, consider the first term in the right-hand side of (6.1). We have

$$\begin{aligned}\left|\sum_{n=1}^{K} f_n^*(\lambda)\overline{f_n^*(\lambda')}\right| &\geq \sum_{n=1}^{K} |f_n^*(\lambda)|^2 - \left|\sum_{n=1}^{K} f_n^*(\lambda)(\overline{f_n^*(\lambda') - f_n^*(\lambda)})\right| \\ &\geq 1 - \left|\sum_{n=1}^{K} f_n^*(\lambda)(\overline{f_n^*(\lambda') - f_n^*(\lambda)})\right|\,,\end{aligned} \tag{6.3}$$

due to the fact that $f_1^*(\lambda) \equiv 1$. Using the inequality $|\lambda' - \lambda| < 2\delta$, which holds since $\lambda, \lambda' \in V_\delta(\mu)$, one may write

$$\left|\sum_{n=1}^{K} f_n^*(\lambda)(\overline{f_n^*(\lambda') - f_n^*(\lambda)})\right| \leq \max_{1\leq n\leq K} \omega_n(2\delta) \sum_{n=1}^{K} |f_n^*(\lambda)|\,,$$

where

$$\omega_n(2\delta) = \sup_{\substack{|\lambda'-\lambda|<2\delta \\ \lambda',\,\lambda\in I}} |f_n^*(\lambda') - f_n^*(\lambda)|$$

is the modulus of continuity of $f_n^*(\lambda)$ on I. Thus, by the uniform continuity of $\{f_n^*(\lambda)\}$ and its uniform boundedness, which follows from the uniform continuity and the boundedness for a fixed λ, one may take δ sufficiently small so that

$$\left|\sum_{n=1}^{K} f_n^*(\lambda)(\overline{f_n^*(\lambda') - f_n^*(\lambda)})\right| < \frac{1}{2}\,. \tag{6.4}$$

From (6.1), (6.2), (6.3) and (6.4), we conclude that

$$(f^*(\lambda), f^*(\lambda'))_{l^2(\mathbb{N})} > 1 - \frac{1}{2} - \frac{1}{2} = 0\,.$$

If λ and λ' are in $\sigma_{pp}(J)$, it must be that $f^*(\lambda) \perp f^*(\lambda')$ since $J = J^*$. This is in contradiction with the above inequality and thus with the accumulation of eigenvalues of J to μ. □

We conclude this section with the following comment. The proof of the spectrum's discreteness crucially relies not only on the uniform asymptotics of $\{f_n^-(\lambda)\}$, but also on the uniform continuity of the elements of $\{f_n^*(\lambda)\}$. Corollary 5.2 allows to properly "connect" this two solutions and take advantage of

the properties of each sequence. Note that this "proper connection" is possible due to the fact that the asymptotic expansion of the decaying solution is uniformly separated from zero for $\lambda \in I \cap \sigma_{pp}(J)$ and all fixed sufficiently large $n \in \mathbb{N}$.

Appendix

A. Asymptotics of the Poincaré coefficients

Let us study the asymptotic behavior of the sequences $\{F_n(\lambda)\}$ and $\{G_n(\lambda)\}$, given by (2.6), as $n \to \infty$. We assume that the sequences $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ given by (2.1) with (2.2) and (2.5). Let I be any bounded interval of $\mathbb{R}_+$. Take a natural number $N \geq \frac{(\sup I)^{1/\alpha}}{2}$. As the definition of $F_n(\lambda)$ requires, we have $q_{2n} \notin I$ as soon as $n \geq N$.

Let us substitute the expression for $\{q_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ given by (2.1) into (2.6). We then write

$$F_n(\lambda) = \frac{c_2}{c_1} A_n(\lambda) - \frac{1}{c_1 c_2} B_n(\lambda) + \frac{c_1}{c_2} C_n\,, \qquad \lambda \in I\,, \quad n \geq N\,,$$

where, for any $\lambda \in I$, $n \geq N$

$$\begin{aligned} A_n(\lambda) &:= \frac{((2n+2)^\alpha - \lambda)(2n)^{2\alpha}}{((2n)^\alpha - \lambda)(2n+1)^\alpha (2n+2)^\alpha} \\ B_n(\lambda) &:= \frac{((2n+1)^\alpha - \lambda)((2n+2)^\alpha - \lambda)}{(2n+1)^\alpha (2n+2)^\alpha} \\ C_n &:= \frac{(2n+1)^\alpha}{(2n+2)^\alpha} \end{aligned}$$

First we obtain asymptotic formulae for $\{A_n(\lambda)\}$, $\{B_n(\lambda)\}$, and $\{C_n\}$ separately.

Since λ is confined in the finite interval I, we have the following *uniform* asymptotic expansion

$$\frac{1}{(2n)^\alpha - \lambda} = \frac{1}{(2n)^\alpha}\left(1 + \frac{\lambda}{(2n)^\alpha} + \frac{\lambda^2}{(2n)^{2\alpha}} + \widetilde{O}_I(n^{-3\alpha})\right)\,, \qquad \text{as} \quad n \to \infty\,.$$

Using the fact that $\frac{(2n)^{2\alpha}}{(2n+1)^\alpha (2n+2)^\alpha} = 1 - \frac{3\alpha}{2n} + O(n^{-2})$ as $n \to \infty$, one obtains

$$A_n(\lambda) = \frac{(2n+2)^\alpha - \lambda}{(2n)^\alpha}\left(1 + \frac{\lambda}{(2n)^\alpha} + \frac{\lambda^2}{(2n)^{2\alpha}} + \widetilde{O}_I(n^{-3\alpha})\right)\left(1 - \frac{3\alpha}{2n} + \widetilde{O}_I(n^{-2})\right)$$

Hence, taking into account (2.2), one has

$$A_n(\lambda) = 1 - \frac{\alpha}{2n} + \widetilde{O}_I(n^{-1-\alpha})\,, \qquad \text{as} \quad n \to \infty\,.$$

By observing that $B_n(\lambda) = A_n(\lambda)\left(\frac{(2n+1)^\alpha-\lambda}{(2n)^\alpha}\right)\left(1-\frac{\lambda}{(2n)^\alpha}\right)$, one easily verifies that

$$B_n(\lambda) = 1 - \frac{2\lambda}{(2n)^\alpha} + \frac{\lambda^2}{(2n)^{2\alpha}} + \widetilde{O}_I(n^{-1-\alpha}), \qquad \text{as} \quad n \to \infty .$$

Clearly,

$$C_n(\lambda) = 1 - \frac{\alpha}{2n} + O(n^{-2}), \qquad \text{as} \quad n \to \infty .$$

Thus, in view of (2.1) and (2.5), one concludes that, as $n \to \infty$,

$$F_n(\lambda) = 2 + \frac{2^{1-\alpha}}{c_1c_2}\lambda n^{-\alpha} - \frac{2^{-2\alpha}}{c_1c_2}\lambda^2 n^{-2\alpha} - \frac{\alpha}{n}\left(1+\frac{1}{2c_1c_2}\right) + \widetilde{O}_I(n^{-1-\alpha}). \quad \text{(A.1)}$$

From the definition of $G_n(\lambda)$ found in (2.6), after substituting the expressions for for $\{q_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ given by (2.1), one can easily verify that

$$G_n(\lambda) = A_n(\lambda)\frac{(2n-1)^\alpha}{(2n)^\alpha} .$$

Thus

$$G_n(\lambda) = 1 - \frac{\alpha}{n} + \widetilde{O}_I(n^{-1-\alpha}) \qquad \text{as} \quad n \to \infty . \qquad \text{(A.2)}$$

B. Asymptotics of $\beta_n(\lambda)$

As in Appendix A we assume that the sequences $\{q_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are defined by (2.1) with (2.2), (2.5) and I is any bounded interval of $\mathbb{R}_+$. Thus the uniform asymptotic expansions (A.1) and (A.2) hold true. By simple algebraic manipulations, one has

$$\begin{aligned} F_n(\lambda)F_{n-1}(\lambda) =& 4 + \frac{2^{3-\alpha}}{c_1c_2}\lambda n^{-\alpha} + \frac{2^{2-2\alpha}}{c_1c_2}\left(\frac{1}{c_1c_2}-1\right)\lambda^2 n^{-2\alpha} \\ & - \frac{4\alpha}{n}\left(1+\frac{1}{2c_1c_2}\right) - \frac{2^{2-3\alpha}}{(c_1c_2)^2}\lambda^3 n^{-3\alpha} + \widetilde{O}_I(n^{-1-\alpha}), \end{aligned} \qquad \text{(B.1)}$$

as $n \to \infty$. On the basis of this result we find the asymptotic formula of $4/(F_n(\lambda)F_{n-1}(\lambda))$. Thus we pass on to the following expansion as $n \to \infty$

$$\begin{aligned} \frac{4}{F_n(\lambda)F_{n-1}(\lambda)} =& 1 + \gamma_1(\lambda)n^{-\alpha} + \gamma_2(\lambda)n^{-2\alpha} + \gamma_3(\lambda)n^{-1} \\ & + \gamma_4(\lambda)n^{-3\alpha} + \widetilde{O}_I(n^{-1-\alpha}), \end{aligned} \qquad \text{(B.2)}$$

where $\gamma_1(\lambda)$, $\gamma_2(\lambda)$, $\gamma_3(\lambda)$, and $\gamma_4(\lambda)$ can be calculated explicitly by multiplying this last expression by $F_n(\lambda)F_{n-1}(\lambda)$ and taking into account (B.1). Thus, for

(B.2) to be true, it must be that

$$\gamma_1(\lambda) = -\frac{2^{1-\alpha}}{c_1c_2}\lambda\,, \qquad \gamma_2(\lambda) = \frac{2^{-2\alpha}}{c_1c_2}\left(\frac{3}{c_1c_2}+1\right)\lambda^2\,,$$
$$\gamma_3(\lambda) = \alpha\left(1+\frac{1}{2c_1c_2}\right)\,, \qquad \gamma_4(\lambda) = -\frac{2^{-3\alpha}}{(c_2c_2)^2}\left(3+\frac{4}{c_1c_2}\right)\lambda^3\,.$$

It remains to multiply (B.2) by $G_n(\lambda)$, using (A.2), and subtract 1 to conclude that the uniform asymptotic expansion as $n\to\infty$ of $\{\beta_n(\lambda)\}$, given in (2.12), is

$$\begin{aligned}\beta_n(\lambda) = &-\frac{2^{1-\alpha}}{c_1c_2}\lambda n^{-\alpha} + \frac{2^{-2\alpha}}{c_1c_2}\left(\frac{3}{c_1c_2}+1\right)\lambda^2 n^{-2\alpha} + \frac{\alpha}{2c_1c_2}n^{-1} \\ &- \frac{2^{-3\alpha}}{(c_1c_2)^2}\left(3+\frac{4}{c_1c_2}\right)\lambda^3 n^{-3a} + \widetilde{O}_I(n^{-1-\alpha})\,.\end{aligned} \tag{B.3}$$

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